Assignment 6: Reducibility and Oracles: Solutions

1. Recall that the strengthened Parameter theorem says that for $n, m > 0$,

$$S^n_m(u_1, \ldots u_n, y) = S^n_m(u'_1, \ldots u'_n, y)$$

implies that $u_1 = u'_1, \ldots, u_n = u'_n$.

(a) Show, by giving a counter-example, that the statement for any $n, m > 0$,

$$S^n_m(u_1, \ldots u_n, y) = S^n_m(u'_1, \ldots u'_n, y')$$

implies that $u_1 = u'_1, \ldots u_n = u'_n, y = y'$ is *not* true.

(b) Under what conditions on y and y' is it true?

(a) Let p be the number of the program
$X_2 \leftarrow X_2 + 1$
$Y \leftarrow Y + 1$
and $\bar{p}$ the number of the program
$Y \leftarrow Y + 1$.
Then both $S^1_1(0, p)$ and $S^1_1(1, \bar{p})$ give the same program:
$X_2 \leftarrow X_2 + 1$
$Y \leftarrow Y + 1$
Thus, we have $S^1_1(0, p) = S^1_1(1, p)$ but $0 \neq 1$ and $p \neq \bar{p}$.

(b) The statement will be true so long as the programs for p and $\bar{p}$ have no statements of the form $X_i \leftarrow X_i + 1$, $i > n$, at the front of their program. (Of course, this is a minor restriction, as any well-written program should not use the input variables other than $X_1, \ldots, X_n$: any temporary variables should be Z_i's.)

2. Prove that $K \leq_1$ FIN.

Let p be the number of the program

(A) IF STP(X_2, X_2, X_1) = 0 GOTO E
$Z \leftarrow Z + 1$
(B) IF $Z \neq 0$ GOTO B

Suppose that X_2 is in K. Then there is some number of steps t_0 after which X_2, run on input X_2, halts, so STP(X_2, X_2, t) will be true for all $t \geq t_0$. Thus, p will be defined exactly for $X_1 < t$, so it will be defined for finitely many t. Conversely, if X_2 is not in K, then the program will always halt, and hence be defined for every X_1. Thus we have

$$\begin{aligned} x \in K \quad &\Leftrightarrow \quad \Phi(z, x, p) \text{ defined for finitely many } z \\ &\Leftrightarrow \quad \Phi(z, S^1_1(x, p)) \text{ defined for finitely many } z \end{aligned}$$

$$\Leftrightarrow \quad S_1^1(x,p) \in \text{FIN}$$

If we define $g(x) = S_1^1(x,p)$, then by the strengthened parameter theorem, g is 1-1 and computable, so $K \leq_1$ FIN.

3. For any sets $D, E \subseteq \mathcal{N}$, define

$$D \oplus E := \{2x : x \in D\} \cup \{2x+1 : x \in E\}$$

Now, suppose that $A \subseteq \mathcal{N}$, with $K \leq_t A$, and define

$$C = \{x \in K : \Phi(x,x) \notin A \oplus \bar{A}\}.$$

Prove that:

(a) $C \leq_t A$,

(b) $A \leq_1 C$.

(a) K being A-computable means that the function C_K is A-computable. Then the following A-program computes C:

`IF` $C_K = 0$ `GOTO E`
`IF` $\Phi(x,x)$ `is even` $\wedge$ $O(\Phi(x,x)/2) = 1$ `GOTO E`
`IF` $\Phi(x,x)$ `is odd` $\wedge$ $O(\phi(x,x) - 1/2) = 0$ `GOTO E`
$Y \leftarrow Y + 1$
Indeed, the above program first tests whether X is in K. If it is even, in then tests whether it is in A, if it is odd, it tests whether it is not in A. If it passes all these tests, then it is in C, so it returns 1.

(b) Let p be the number of the program

$Y \leftarrow 2X_2 + 1$

If $X_2 \in A$, then the output $Y \notin A \oplus \bar{A}$, and if $X_2 \notin A$, then $Y \in A \oplus \bar{A}$. So, for any z,

$$x \in A \Leftrightarrow \Phi(z,x,p) \notin A \oplus \bar{A}.$$

In particular, if we let $z = S_1^1(x,p)$, then we get

$$\begin{aligned} x \in A &\Leftrightarrow \Phi(S_1^1(x,p),x,p)) \notin A \oplus \bar{A} \\ &\Leftrightarrow \Phi(S_1^1(x,p),S_1^1(x,p)) \notin A \oplus \bar{A} \\ &\Leftrightarrow S_1^1(x,p) \in C \end{aligned}$$

Thus, if we let $g(x) = S_1^1(x,p)$, then by the strengthened parameter theorem, g is 1-1 and computable, so $A \leq_1 C$.

4. For any total function g, show that if B and C are g-R.E., then so are $B \cup C$ and $B \cap C$.

   Suppose B and C are g-RE, say by partially g-computable functions h and k. Then the following program is g-partially computable:

   $Y \leftarrow h(X)$
   $Y \leftarrow k(X)$
   and halts exactly when X is in both B and C. Thus $B \cap C$ is g-RE.

   Let h and k be partially computed by g-programs with numbers p and q. Then by the relativized step-counter theorem, the following is a g-program:

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   (A) IF STP_g(X,p,t) GOTO E
   IF STP_g(X,q,t) GOTO E
   t ← t + 1
   GOTO A
   ```
   and halts if either p or q halts on input X. Thus, $B \cup C$ is g-RE.

5. For sets A, B, C does A being B-R.E., and B being C-R.E. imply A is C-R.E? Either prove or give a counter-example.

   Note that $\bar{K}$ is K-recursive, and hence K-RE. Also, K is 1-RE. But, we know $\bar{K}$ is not 1-RE (if it was, we would have that K is recursive, which is a contradition). Thus, the statement is false.