Assignment 1: Proofs and Simple Programs: Solutions

1. If $\mathcal{E}$ is the set of even numbers $(0, 2, 4, 6...)$, and $\mathcal{N}$ the set of natural numbers $(0, 1, 2, 3...)$, give examples of:

   (a) A 1-1 function from $\mathcal{N}$ to $\mathcal{E}$,

   (b) An onto function from $\mathcal{E}$ to $\mathcal{N}$.

   Be sure to *prove* that the first function is 1-1, and the second function is onto!

   (a) We claim that the function $f : \mathcal{N}$ to $\mathcal{E}$, defined by $f(n) = 2n$ is 1-1. First, note that the function is well-defined since $2n$ is always an even number. To prove that it is 1-1, suppose that there are natural numbers a, b such that $f(a) = f(b)$. By the definition of f, $2a = 2b$. Dividing by 2 on both sides gives $a = b$. Thus, $f(a) = f(b)$ implies $a = b$, so f is 1-1.

   (b) We claim that the function $f : \mathcal{E}$ to $\mathcal{N}$, defined by $f(n) = f(n/2)$, is onto. First, note the function is well-defined (that is, $f(n)$ is a natural number), since an even number divided by 2 is always a natural number. To show that it is onto, suppose that a is an arbitrary natural number. Then $2a$ is an even number, and $f(2a) = 2a/2 = a$. So, there is a element of the domain which f maps to a. Thus, f is onto.

2. Prove that the composite of two isomorphisms is an isomorphism.

   Suppose that $f : X \to Y$ and $g : Y \to Z$ are two isomorphisms. That is, f is 1-1 and onto, and g is 1-1 and onto. We must prove that the composite function gf is 1-1 and onto. To prove that it is 1-1, suppose that $gf(x) = gf(y)$. This means $g(f(x)) = g(f(y))$. Since g is 1-1, $f(x) = f(y)$. Since f is 1-1, this gives $x = y$. Thus, $gf(a) = gf(b)$ implies $a = b$, so gf is 1-1.

   To prove that gf is onto, suppose that $z \in Z$. Since g is onto, there exists a $y \in Y$ such that $g(y) = z$. Since f is onto, there exists an x such that $f(x) = g(y)$. Then $gf(x) = g(f(x)) = g(y) = z$. So, there is an element of the domain which gf maps to z. Thus, gf is onto. Since gf is 1-1 and onto, gf is an isomorphism. Thus, the composite of any two isomorphisms is an isomorphism.

3. Use strong induction to prove that any amount of postage of 12 cents or more can be achieved by using some combination of 4 and 5 cent stamps.

   We will prove the result by strong induction on $n \geq 12$. We will begin with four base cases:

(a) if $n = 12$, then we can use 3 4-cent stamps,

(b) if $n = 13$, we can use 2 4-cent stamps and 1 5-cent stamp,

(c) if $n = 14$, we can use 1 4-cent stamp and 2 5-cent stamps,

(d) if $n = 15$, we can use 3 5-cent stamps.

Thus, the result is proven for $12 \leq n \leq 15$. Induction hypothesis: assume the result is true for all $12 \leq k < n$, where $n \geq 16$. We need to prove the result for n. Since $n \geq 16$, we know that $n - 4 \geq 12$. Thus, by the induction hypothesis, we can use some combination of 4 and 5-cent stamps to achieve a postage of $n - 4$. If we add a 4-cent stamp to this combination, we will have a postage of n. Thus, the result is true for n.

Thus, by strong induction, the result is true for all $n \geq 12$.

4. Write a program in the language $\mathcal{P}$ that returns 1 if the input is odd, and 0 if the input is even (without using macros).

The idea of the following program is to decrease the variable X_1 until it reaches 0. The variable Y will alternate between 0 and 1 as X_1 decreases. It initially starts at 1, since 0 is even. After one decrease of X_1, Y decreases. After another decrease of X_1, Y increases. It then continues to alternate, increasing and decreasing, so that once X_1 becomes 0, it is 1 is X_1 was even, and 0 if it was odd. The program:

$Y \leftarrow Y + 1$
(A) IF $X_1 \neq 0$ GOTO A
$Z_1 \leftarrow Z_1 + 1$
IF $Z_1 \neq 0$, GOTO E
(B) $X_1 \leftarrow X_1 - 1$
$Y \leftarrow Y - 1$
IF $X_1 \neq 0$ GOTO C
$Z_2 \leftarrow Z_2 + 1$
IF $Z_2 \neq 0$ GOTO E
(C) $X \leftarrow X - 1$
$Y \leftarrow Y + 1$
$Z_3 \leftarrow Z_3 + 1$
IF $Z_3 \neq 0$ GOTO A (E)

5. Write a program in the language $\mathcal{P}$ that computes the function $f(x) =$ the greatest natural number n such that $n^2 \leq x$. You may use any macros we have discussed in class.

The idea of the following program is to continually increase Y until Y^2 is strictly greater than the input X_1. To calculate Y^2, we use the multiplication macro. To determine whether Y^2 is strictly greater than X_1, we put them into local variables Z_1 and Z_2, then decrease these variables until one of them is 0. If Z_1 is 0, we increase Y and try again. If Z_2 is 0 and Z_1 is not, then we know that X_1 is stricly greater than Y^2.

At that point, we reduce Y by 1 and return that value, as that is the largest n such that $n^2 \leq X_1$.

Finally, the first two lines of the program just skip to the end and return 0 if X_1 is 0.

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IF X_1 ≠ 0 GOTO A
GOTO E
(A) Y ← Y + 1
Z_1 ← Y · Y
Z_2 ← X_1
(B) IF Z_1 ≠ 0 GOTO C
GOTO A
(C) Z_1 ← Z_1 − 1
Z_2 ← Z_2 − 1
IF Z_2 ≠ 0 GOTO B
GOTO D
(D) Y ← Y − 1
GOTO E
(E)
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