Quiz 4 Solution

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1.(a). Consider T = 100.
And there are 3 items:

	G1	G2	G3
Profit	100	70	60
Weight	70	50	40

The greedy solution takes G1 which does not give a maximum profit. The highest profit can be gained by taking G2 and G3.

(b). Assume $w_1, w_2, \ldots, w_n$, T are strictly positive integers. Define m[i , w] to be the maximum value that can be attained with weight less than or equal to w using goods up to i.
We can define m[i , w] recursively as follows:

- m[0, w] = 0
- m[i , 0] = 0
- m[i , w] = m[i - 1, w] if $w_i > w$ (the new good is more than the current weight limit)
- m[i , w] = max (m[i - 1, w] , m[i - 1, w - w_i] + p_i) if $w_i \leq w$

The solution can then be found by calculating m[n , T]. To do this efficiently we can use a table to store previous computations. This solution will therefore run in O(nT) time and O(nT) space.

So, having this algorithm:

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Init m[0..n, 0..T]

For w = 0 to w = T
    m[0, w] = 0

For i = 0 to i = n
    m[i, 0] = 0

For i = 1 to i = n
    For w = 1 to w = T
        If (wi > w) then m[i , w] = m[i - 1, w]
        Else
         m[i , w] = max (m[i - 1, w] , m[i - 1, w - wi] + pi)
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The answer is in m[n,T].

(c). We use induction to prove this algorithm:

base: n = 0: m[0, T] = 0 (There is no good to take)

n = 1: m[1, T] = m[0, T] = 0 if $w_1 > T$ else

= max (m[0, T] , m[0, T- w_1] + p_1) = max (0, 0 + p_1) = p_1

which is optimal.

Now, assume algorithm is optimal for n goods. i.e. m[i , w] contains the optimal answer for $1 \le i \le n$, and $1 \le w \le T$.

We want to prove that it is optimal for n + 1 goods, as well. Which means m[n+1 , T] contains the optimal answer:

According to our definition of m[i , w];

If $w_{n+1} > T$ then m[n+1 , T] = m[n , T] which is correct.

But, if $w_{n+1} \le T$ then m[n+1 , T] = max (m[n, T] , m[n, T- w_{n+1}] + p_{n+1})

Obviously, we have only two options for n+1[st] good:

1) whether to include it in answer. So, m[n+1 , T] = m[n, T - w_{n+1}] + p_{n+1}.
2) or not. So, m[n+1 , T] = m[n, T].

And, in both cases m[n, T - w_{n+1}] and m[n, T] contain optimal answers, in accordance with our assumption. Since we take the maximum of these two values, m[n+1 , T] will contain optimal answer as well.