Solution to Quiz 2

Fatemeh

8 October 2010

Let denote the n items with $i_1, \ldots, i_n$. Each item i_k, has three parameters assigned to it, $(c_{i_k}, p_{i_k}, w_{i_k})$ where c_{i_k} denotes the time required for constructing the item, p_{i_k} for painting and w_{i_k} for wrapping it. According to the problem description, items can not be constuced concurrently but the painting and wrapping processes can happen in parallel for different items. So, we re-write the tuple assigned to item i_k as (c_{i_k}, pw_{i_k}) where $pw_{i_k} = p_{i_k} + w_{i_k}$ denotes the post-construction time for item i_k. Also note that once we line up the items on the construction line, we can assign e new parameter to each item that denotes the total time that will take the item to complete all the three processes with respect to where it stands in the line; $t_{i_k} = \sum_{j<k} c_{i_j} + c_{i_k} + p_{i_k} + w_{i_k} = \sum_{j<k} c_{i_j} + c_{i_k} + pw_{i_k}$. Here we assume that the items are lined up according to how they initialy were indexed. The goal is to minimize the maximum value that t_k takes with respect to the arrangment of the construction line, i.e. minimizing $T = max\{t_{i_k}, 1 \leq k \leq n\}$.

$Algo_1$:

1. Order the items decreasingly with respect to $pw_{i_k}, \quad 1 \leq k \leq n$.
2. The item with the maximum post-construction time goes first in the construction line.
3. Remove this item and repeat step 2 for the rest of the items.

Claime: $Algo_1$ is an optimal one.
Proof: Let $Algo_{opt}$ be another algorithm with an optimal T_{opt} that arranges items as $S = \{i_{o_1}, \ldots, i_{o_n}\}$. Using an "exchange" argument we show that by rearranging $Algo_{opt}$ according to $Algo_1$, the value of T_{opt} won't increase.

Case 1: If for every $1 \leq k \leq n$, $pw_{i_{ok}} \geq pw_{i_{ok+1}}$, then the two algorithms are suggesting a similar arrangement.

Case 2: Let j be an index for which

$$pw_{i_{oj}} < pw_{i_{oj+1}}. \tag{1}$$

We show that swapping the two items in the construction line won't increase T_{opt}, and hence by swapping any pair of items that have a similar property as Equation (1), we will end up rearranging the output of $Algo_{opt}$ according to $Algo_1$ while its optimality is preserved.

Let $T' = max\{t_{i_{oj}}, t_{i_{oj+1}}\}$. If we show that T' won't increase after swapping the two items, we conclude that so doesn't T_{opt}. (Note that $t_{v_{ow}}$ won't change for other items before and after swapping

i_{oj} and i_{oj+1}.) To investigate the possible cases, we use an index 1 to refer to the arrangment before swapping the two items and index 2 indicates the parameters after swapping.

$$\begin{cases} t^1_{i_{oj}} = \sum_{k<j} c_{i_{ok}} + c_{i_{oj}} + pw_{i_{oj}} \\ t^1_{i_{oj+1}} = \sum_{k<j} c_{i_{ok}} + c_{i_{oj}} + c_{i_{oj+1}} + pw_{i_{oj+1}} \end{cases} \qquad \begin{cases} t^2_{i_{oj}} = \sum_{k<j} c_{i_{ok}} + c_{i_{oj+1}} + c_{i_{oj}} + pw_{i_{oj}} \\ t^2_{i_{oj+1}} = \sum_{k<j} c_{i_{ok}} + c_{i_{oj+1}} + pw_{i_{oj+1}} \end{cases}$$

$$T_1' = t^1_{i_{oj+1}} \qquad T_2' : \textit{ can be either one}$$

Now by simple math we can conclude that for any possible valuee of T_2', it always holds that $T_1' \geq T_2'$. This completes the proof of the Claim. □